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Many-Valued Logic

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1 When is a Logic Many-Valued?

The most natural and straightforward step towards the construction of a many-valued logic is to introduce logical values next to truth and falsity. Thereby, one has to reject the principle of bivalence, that every proposition has exactly one of the two logical values. Another, indirect way consists in challenging the classical laws concerning the sentence connectives and introducing non-truth-functional connectives into the language, among them the modal connectives of possibility and necessity. In either case the semantics adequate is different from the classical, that is Boolean, thus the logic under consideration is non-classical.

2 Roots, Motivations, and Early History

The roots of many-valued logics can be traced back to Aristotle (fourth century BC) who considered, within the modal framework, *future contingents* sentences. In Chapter IX of *De Interpretatione* Aristotle provides the time-honored sentence-example representing this category: 'There will be a sea-battle tomorrow.' The Philosopher from Stagira emphasizes the fact that future contingents are neither actually true nor actually false, which suggests the existence of the 'third' logical status of propositions.

The prehistory of many-valued logic falls on the Middle Ages. More serious attempts to create non-classical logical constructions, three-valued mainly, appeared only on the turn of the nineteenth century. The evaluation to what extent these different approaches by Duns Scott, William Ockham, Peter de Rivo and Hugh MacColl, Charles S. Peirce, Nicolai A.Vasil'ev were important for the topic is not easy. In most cases the division of the totality of propositions into three categories was supported by some considerations dealing with some modal or temporal concepts. Eventually, some criteria of the distinction were applied and the propositions mostly were grouped as either 'affirmative,' 'negative,' or 'indifferent.'

Philosophical motivations for logical many-valuedness may roughly be classified as *ontological* and *epistemic*. First of them focus on the nature of objects and facts, while the others refer the knowledge status of actual propositions. The 'Era of many-valuedness' was finally inaugurated in 1920 by Łukasiewicz (1920) and Post (1920). The thoroughly successful formulations of many-valued logical constructions were possible in the result of an adaptation of the truth table method applied to the classical logic by Frege in 1879, Peirce in 1885 and others. The impetus thus given bore the Łukasiewicz and Post method of logical algebras and matrices. Apparently different proposals of the two scholars had quite different supports.

3 Łukasiewicz Three-Valuedness

Though 1920 is the year of publication of Łukasiewicz's article in an official journal *Ruch Filozoficzny* his finding was published as soon as March 7, 1918. In that paper Łukasiewicz enriched the set of the classical logical values 0 and 1 with an intermediate value 1/2 and laid down the principles of his calculus referring to Aristotle's argument. His *future contingent* proposition read "I shall be in Warsaw at noon on 21 December of the next year."

First Łukasiewicz's interpretation of the third logical value 1/2 was as a 'possibility' or 'indeterminacy.' Accordingly, the interpretation of negation and implication has been extended in the following tables:

x	$\neg x$
0	1
1/2	1/2
1	0

$\to$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

(the truth tables of binary connectives $*$ are viewed as follows: the value of α is placed in the first vertical line, the value of β in the first horizontal line and the value of $\alpha * \beta$ at the intersection of the two lines).

The remaining standard connectives introduced through definitions

$$\alpha \vee \beta = (\alpha \to \beta) \to \beta$$
$$\alpha \wedge \beta = \neg (\neg\alpha \vee \neg\beta)$$
$$\alpha \equiv \beta = (\alpha \to \beta) \wedge (\beta \to \alpha).$$

have the tables:

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\equiv$	0	1/2	1
0	1	1/2	0
1/2	1/2	1	1/2
1	0	1/2	1

A valuation of formulas in Łukasiewicz three-valued logic is any function v: *For* $\to$ $\{0, 1/2, 1\}$ of the set of all formulas *For* compatible with the above tables. A *tautology* is a formula which under any valuation v takes on the *designated* value 1.

The set $Ł_3$ of tautologies of three-valued logic of Łukasiewicz differs from *TAUT*. So, for instance, neither the law of the excluded middle, nor the principle of contradiction is in $Ł_3$. To see this, it suffices to assign 1/2 for p: any such valuation also associates 1/2 with *EM* and *CP*. The thorough-going refutation of these two laws was intended to codify the principles of indeterminism.

Another property of new semantics is that some classically inconsistent formulas are no more contradictory in $Ł_3$. One such formula:

(*) $p \equiv \neg p$,

is connected with the famous Russell paradox of the 'set of all sets that are not their own elements.' *Russell's set* is defined by the equation

$$Z = \{x : x \notin x\}.$$

And the resulting paradox

$$Z \in Z \equiv Z \notin Z,$$

is a substitution of (*). Russell paradox ceases to be antinomy in $Ł_3$ since putting 1/2 for p makes the formula true and therefore (*) is non-contradictory. Łukasiewicz found it a strong argument in favor of his three-valued logic.

4 Post Logics

Post's proposal was made on the margin of the completeness proof of the classical logic. It consists in defining n-valued (n finite) 'logic algebras' saving the classical property of functional completeness of the set of connectives (the property permits the definition of all other possible connectives), cf. Post (1920, 1921).

Following *Principia Mathematica* Post takes the negation ($\neg$) and disjunction ($\vee$) connectives as primitive. For any natural $n \geq 2$ he considers a linearly ordered set

$$Pn = \{ t_1, t_2, \ldots, t_n \},$$

$t_n < t_j$ iff $i < j$, with two operations: unary *rotation* (or *cyclic negation*) $\neg$ and binary *disjunction* $\vee$, where

$$\neg t_i = \begin{cases} t_{i+1} \text{ if } i \neq n \\ t_1 \text{ if } i = n \end{cases} \qquad t_i \vee t_j = t_{imax\{i,j\}}$$

Thus, for example for $n = 4$ the truth tables of these connectives are the following:

x	$\neg x$
t_1	t_2
t_2	t_3
t_3	t_4
t_4	t_1

$\vee$	t_1	t_2	t_3	t_4
t_1	t_1	t_2	t_3	t_4
t_2	t_2	t_2	t_3	t_4
t_3	t_3	t_3	t_3	t_4
t_4	t_4	t_4	t_4	t_4

It is easy to see that for $n = 2$ Post logic coincides with the negation–disjunction version of the classical logic: the set $P_2 = \{ t_1, t_2\}$ may be identified as containing 0 and 1, respectively, and then the Post negation and disjunction are isomorphic variants of the classical connectives on P_2. The relation to *CPC* breaks for $n > 2$. In all these cases truth tables of negation are not compatible with the classical one is due to the fact that t_1 always corresponds to 0 and t_n to 1. Though $\neg t_n = t_1$, $\neg t_1$ equals t_2 and thus then is not t_n.

Post considers t_n as the distinguished value. Among special laws of all its logics ($n > 2$) the following many-valued counterpart of the classical law of the excluded middle

$$p \vee \neg p \vee \neg\neg p \vee \ldots \underbrace{\neg\neg \ldots \neg}_{(n-1)\text{ times}} p.$$

deserves attention. The absence of the counterparts of some other classical tautologies follows directly from the properties of negation.

The most important property of Post algebras is their functional completeness: by means of the two primitive functions, every finite-argument function on P_n can be defined. In particular, then, the constant functions may also be defined and hence the 'logical values' $t_1, t_2, \ldots, t_n$.

Post suggests interpreting the elements of P_n as $(n - 1)$-element-tuples $P = (p_1, p_2, \ldots, p_{n-1})$ of ordinary two-valued propositions $p_1, p_2, \ldots, p_{n-1}$ subject to the condition that the true propositions are listed before the false. Then

($\neg$) $\neg P$ if formed by replacing the first false element by its denial, otherwise it is a sequence of false propositions.

($\vee$) When $P = (p_1, p_2, \ldots, p_{n-1})$ and $Q = (q_1, q_2, \ldots, q_{n-1})$, then $P \vee Q = (p_1 \vee q_1, p_2 \vee q_2, \ldots, p_{n-1} \vee q_{n-1})$.

For n = 4 one gets the following 3-tuples:

(0, 0, 0)	t_1
(1, 0, 0)	t_2
(1, 1, 0)	t_3
(1, 1, 1)	t_4.

This interpretation shows that the values in different Post logics should be understood differently.

5 Łukasiewicz Logics

In 1922 Łukasiewicz generalized his three-valued logic and defined the family of many-valued logics, both finite and infinite-valued, see Łukasiewicz (1970: 140). The set of logical values of n-valued logic for any natural $n \geq 2$ is

$$L_n = \{0,\ 1/(n-1),\ 2/(n-1),\ \ldots,\ (n-2)/(n-1),\ 1\}.$$

First infinite logic is based on the set of all fractions in the real interval [0,1],

$$L_{\mathcal{N}0} = \{s/w\colon 0 \leq s \leq w,\ s,\ w \in N \text{ and } w \neq 0\}$$

and the second on the whole interval [0,1], $L_{\mathcal{N}1} = [0,1]$. In all these cases 1 is taken as the only designated value and the connectives are defined as follows:

1. $\neg x = 1 - x$
   $x \rightarrow y = min(1,\ 1 - x + y)$
2. $x \vee y = (x \rightarrow y) \rightarrow y = max(x,\ y)$
   $x \wedge y = \neg(\neg x \vee \neg y) = min(x,\ y)$
   $x \equiv y = (x \rightarrow y) \wedge (y \rightarrow x) = 1 - |x - y|.$

To give an idea of what truth tables of finite valued logics look like, we now show the tables of negation and implication in the five-valued logic of Łukasiewicz:

x	$\neg x$
0	1
1/4	3/4
2/4	2/4
3/4	1/4
1	0

$\rightarrow$	0	1/4	2/4	3/4	1
0	1	1	1	1	1
1/4	3/4	1	1	1	1
2/4	2/4	3/4	1	1	1
1/4	1/4	2/4	3/4	1	1
1	0	1/4	2/4	3/4	1

Łukasiewicz matrices have this exceptional property that in all of them the set {0,1} is closed with respect to all connectives. This together with the fact that the tables for all usual connectives on this set coincide with the classical truth tables yields the fact that the set of all tautologies of every Łukasiewicz logic, $Taut_n$, is a subset of the set of tautologies of the CPC which actually is $Taut_2$. The inclusion

$$Taut_n \subseteq Taut_2$$

extends to the famous Lindenbaum's condition on mutual relations in the family of finite Łukasiewicz logic. Namely, that for any natural n, m (both ≥ 2)

$$Taut_n \subseteq Taut_m \quad \text{if and only if } m - 1 \text{ is a } \textit{divisor} \text{ of } n - 1\ .$$

Infinite Łukasiewicz matrices have the same set of tautologies equal to the intersection of the contents of all finite matrices: $\cap \{ Taut_n: n \geq 2, n \in N\}$.

Contrary to Post none of the Łukasiewicz logics $Ł_n$ $(n \neq 2)$ is functionally complete since no constant function except 0 or 1 is definable. Adding all suitable constants to the stock of connectives makes each finite logic complete. McNaughton (1951) formulated and proved an ingenious definability criterion for Łukasiewicz matrices, both finite and infinite, showing the mathematical beauty of Łukasiewicz's logic constructions.

As early as 1931 Wajsberg gave an axiomatization of $Ł_3$. Taking the rules *MP* and *SUB* he established the four axioms

W1 $p \to (q \to p)$
W2 $(p \to q) \to ((q \to r) \to (p \to r))$
W3 $(\neg p \to \neg q) \to (q \to p)$
W4 $((p \to \neg p) \to p) \to p$.

Since the other Łukasiewicz connectives are definable, the axiomatizability result obviously applies to the whole $Ł_3$. It is worth noting that W1–W4 was the first axiom system of many-valued logics. Still earlier, in 1930, Łukasiewicz conjectured that his $\mathcal{N}_0$-valued logic was axiomatizable (Łukasiewicz and Tarski 1930) by five axioms: W1, W2, and

L3 $((p \to q) \to q)\ ((q \to p) \to p)$
L4 $(\neg p \to \neg q)\ (q \to p)$
L5 $((p \to q)\ (q \to p))\ (q \to p)$.

The response came only in 1958 with two works showing the dependence, and thus, the eliminability of L5. In addition, two further completeness proofs, one syntactic and the other algebraic, were derived see Rose and Rosser (1958) and Chang (1959).

6 Kleene and Bochvar Logics

In 1938 two similar, though independent, three-valued systems of logic were invented by Kleene and Bochvar. The epistemic arguments behind their construction relate to indeterminacy or to meaninglessness.

Kleene's (1938) main assumption is that there are propositions whose logical truth (t) or falsity (f) is either undefined, undetermined by means of accessible algorithms, or is not essential. The third value of undefiniteness (u) is reserved for this category of propositions. Further to that the tables of the standard connectives save the classical behavior towards t and f and looks like:

α	$\neg\alpha$
f	t
u	u
t	f

$\to$	f	u	t
f	t	t	t
u	u	u	t
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	t
t	t	t	t

$\wedge$	f	u	t
f	f	f	f
u	f	u	u
t	f	u	t

$\equiv$	f	u	t
f	t	u	f
u	u	u	u
t	f	u	t

Kleene's logic has no tautologies. This, somewhat striking, feature follows from the fact that any valuation which assigns u to every propositional variable also assigns u to any formula.

In 1952, in his monograph *Introduction to Metamathematics* Kleene refers to the connectives of his 1938 logic as strong and introduces another set of weak connectives: retaining the negation and equivalence he defines the three others by the tables

$\rightarrow$	f	u	t
f	t	u	t
u	u	u	u
t	f	u	t

$\vee$	f	u	t
f	f	u	t
u	u	u	u
t	t	u	t

$\wedge$	f	u	t
f	f	u	f
u	u	u	u
t	f	u	t

The novel truth tables are to describe the employment of logical connectives in respect of those arithmetical propositional functions whose decidability depends on the effective recursive procedures. They are constituted according to the rule of saying that any single appearance of u results in the whole context taking u. The original arithmetic motivation states that indeterminacy occurring at any stage of computation makes the entire procedure undetermined. While the first Kleene logic was made to render the analysis of partially defined propositional functions possible, the second was inspired by the studies within the mathematical theory of recursion, see Kleene (1952).

Bochvar's (1938) approach is directed towards solving paradoxes emerging with the classical logic and set theory based on it. The propositional language of Bochvar logic has two levels corresponding to the object language and to metalanguage. They both contain connectives being counterparts of negation, implication, disjunction, conjunction, and equivalence. The *internal* connectives are conservative generalizations of the classical ones, in the sequel they will be denoted similarly. The *external* connectives are devised to characterize the relationship between logical values of propositions. Both sets are initially described using the values corresponding to two kinds of meaningful sentences that is of truth (t) and falsity (f), and the third value u reserved for meaningless sentences.

The tables of internal connectives have been set according to the rule: 'every compound proposition including at least one meaningless component is meaningless, in other cases its value is determined classically.' Consequently, the internal Bochvar logic coincides with the weak Kleene logic.

The external 'metalinguistic' connectives are supposed to express the predicates '. . . is true' and '. . . is false' and have the following 'meanings':

external nexgation:	$\neg^* \alpha$	'α is false'
external implication:	$\alpha \rightarrow^* \beta$	'if α is true, then β is true'
external disjunction:	$\alpha \vee^* \beta$	'α is true or β is true'
external implication:	$\alpha \wedge^* \beta$	'α is true and β is true'
external implication:	$\alpha \equiv^* \beta$	'α is true iff β is true'

Their truth tables are as follows:

α	$\neg^*\alpha$
f	t
u	t
t	f

$\rightarrow^*$	f	u	t
f	t	t	t
u	t	t	t
t	f	f	t

$\vee^*$	f	u	t
f	f	f	t
u	f	f	t
t	t	t	t

$\wedge^*$	f	u	t
f	f	f	f
u	f	f	f
t	f	f	t

$\equiv^*$	f	u	t
f	t	t	f
u	t	t	f
t	f	f	t

As a result, the external logic is a 'three-valued' version of the classical logic. This is due to the fact that the truth tables of all external connectives 'identify' the values u and f, whereas the behavior of these connectives with regard to f and t is classical.

7 Towards a General Framework

With a view to unification, Rosser and Turquette (1952) established some special *standard conditions* that make finitely many-valued logics resemble the classical propositional logic. This, on a certain level of investigation, permitted the simplification or solving of some metalogical questions, such as axiomatization and the extension to predicate logics.

Assume that $n \geq 2$ is a natural number and $1 \leq k < n$. Let $E_n = \{1, 2, \ldots, n\}$ be the set of logical values and $D_k = \{1, 2, \ldots, k\}$ as the set of designated values. Rosser and Turquette assume that the natural number ordering conveys decreasing degrees of truth. So, 1 always refers to 'truth' and n takes the role of falsity.

Next come the conditions concerning propositional connectives, which have to represent negation ($\neg$), implication ($\rightarrow$), disjunction ($\vee$), conjunction ($\wedge$), equivalence ($\equiv$) and special one-argument connectives $j_1, \ldots, j_n$. The respective connectives satisfy the *standard conditions* if for any $x, y \in E_n$ and $i \in \{1, 2, \ldots, n\}$

$\neg x \in D_k$	if and only if	$x \notin D_k$
$x \rightarrow y \notin D_k$	if and only if	$x \in D_k$ and $y \notin D_k$
$x \vee y \notin D_k$	if and only if	$x \in D_k$ or $y \in D_k$
$x \wedge y \in D_k$	if and only if	$x \in D_k$ and $y \in D_k$
$x \equiv y \in D_k$	if and only if	either $x, y \in D_k$ or $x, y \notin D_k$
$j_i(x) \in D_k$	if and only if	$x = i$.

Any many-valued logic $L_{n,k}$ having standard connectives as primitive or definable is called *standard*.

The class of many-valued logics, whose connectives fulfill standard conditions is quite large. It contains, 'obviously,' all Post logics since they are functionally complete. All finite Łukasiewicz logics are also standard; note that the mapping $f(x) = n - (n - 1)$ transposes the original values $\{0, 1/(n-1), 2/(n-1), \ldots, n-2/(n-1), 1\}$ onto $\{1, 2, \ldots, n\}$. A moment's reflection shows that original Łukasiewicz disjunction and conjunction satisfy standard conditions. In turn, the other required connectives including j's, $j_i(x) = 1$ iff $x = i$, are definable.

Using their framework, Rosser and Turquette positively solved the problem of axiomatizability of known systems of many-valued logic, including n-valued Łukasiewicz and Post logics. Actually, any $\{\rightarrow, j_1, j_2, \ldots, j_n\}$ – standard logic $L_{n,k}$ is axiomatizable by means of the rule *MP* and *SUB* and the following set of axioms:

A1 $p \rightarrow (q \rightarrow p)$
A2 $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$
A3 $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$
A4 $(j_i(p) \rightarrow (j_i(p) \rightarrow q)) \rightarrow (j_i(p) \rightarrow q)$
A5 $(j_n(p) \rightarrow q) \rightarrow ((j_{n-1}(p) \rightarrow q) (\ldots \rightarrow ((j_1(p) \rightarrow q) \rightarrow q) \ldots))$
A6 $j_i(p)\, p \rightarrow$ for $i = 1, 2, \ldots, k$
A7 $j_{i(r)}(p_r) \rightarrow (j_{i(r-1)}(p_{r-1}) \rightarrow (\ldots \rightarrow (j_{i(1)}(p_1) \rightarrow j_f(F(p_1, \ldots, p_{r-1}, p_r))) \ldots))$
where $f = f(i(1), \ldots, i(r))$;

symbols F and f in A7 represent, respectively, an arbitrary connective and the function associated with it.

The first three axioms describe the properties of pure classical implication sufficient, among others, to get the deduction theorem in its classical version. The remaining axioms bridge, due to the properties of j connectives and of the implication, the semantic and syntactic properties. Checking the soundness of the axioms is easy and is heavily based on procedures known from classical logic. The completeness proof, however, requires much calculation and involves a complicated induction.

8 On Quantification

Many-valued predicate calculi are usually built along the classical pattern. In that case a first-order language with two standard quantifiers, general $\forall$ and existential $\exists$, are considered. Mostly, the starting point is the substitutional conception of quantifiers according to which $\forall$ and $\exists$ are (infinite) generalizations of conjunction and disjunction, respectively. Accordingly, for a finite domain $U = \{a_1, a_2, \ldots, a_n\}$, the commutative and associative connectives of conjunction ($\wedge$) and disjunction ($\vee$):

$$\forall x F(x) \equiv_U F(a_1) \wedge F(a_2) \wedge \ldots \wedge F(a_n)$$
$$\exists x F(x) \equiv_U F(a_1) \vee F(a_2) \vee \ldots \vee F(a_n)$$

($\equiv_U$ means the equivalence of the formulae at any interpretation in U, $a_1, a_2, \ldots, a_n$ being nominal constants ascribed to the objects of the domain). In finite-valued logical calculi constructed upon linear matrices, quantifiers are defined 'directly' through algebraic functions related to the above-mentioned connectives. Thus, for example, for finite Łukasiewicz and Post logics, for any interpretation f in a domain U

$$f(\forall x f(x)) = min\{f(F(a)) : a \in U\}$$
$$f(\exists x F(x)) = max\{f(F(a)) : a \in U\}.$$

For other calculi, the semantic description of quantifiers may vary. Thus, for example, the clauses defining quantifiers in the first-order Bochvar logic should as follows:

$$f(\forall xF(x)) = \begin{cases} \text{t} & \text{when } f(F(a)) = \text{t for every } a \in U \\ \text{u} & \text{when } f(F(a)) = \text{u for some } a \in U \\ \text{f} & \text{otherwise} \end{cases}$$

$$f(\exists xF(x)) = \begin{cases} \text{f} & \text{when } f(F(a)) = \text{f for every } a \in U \\ \text{u} & \text{when } f(F(a)) = \text{u for some } a \in U \\ \text{t} & \text{otherwise.} \end{cases}$$

Axiomatic systems of many-valued predicate logics are usually built as extensions of axiom systems of the grounds of propositional calculi in a similar way to classical logic, see Rasiowa and Sikorski (1963) and Rasiowa (1974). Proofs of completeness for finitely-valued calculi do not, in general, create difficulties. Axiomatizability of several important calculi of this kind are assured by Rosser and Turquette's result extending the standard condition's approach to quantifiers, see Rosser and Turquette (1952).

Introducing quantifiers to logics with infinitely many values in the semantical plane may be problematic. Thus, for example, applying the above-mentioned procedure to the $\mathcal{N}_0$-valued Łukasiewicz logic is impossible since in the case when U is infinite it may happen that the set $\{ f(F(a)) : a \in U\}$ does not contain the least or the greatest element and therefore *min* and *max* functions cannot be used in the definition. In turn, in the $\mathcal{N}_1$-valued Łukasiewicz logic, the interpretations of quantifiers are introduced provided that for any interpretation in a non-empty domain U

$$f(\forall xF(x)) = inf\{f(F(a)) : a \in U\}$$
$$f(\exists xF(x)) = sup\{f(F(a)) : a \in U\},$$

see Mostowski (1961). However, it appeared that $\mathcal{N}_1$-valued predicate calculus thus obtained is not axiomatizable, Scarpelini (1962). The problem of the completeness of this logic appeared extremely complex and the experience gained while attempting to constitute such a proof raised the so-called *continuous model theory* (see Chang and Keisler 1966).

Rosser and Turquette (1952) invented a general theory of quantification for a class of finitely many-valued logics. Starting from the intuition that ordinary quantifiers are functions on the set of pairs (x, F), where x is a nominal variable and F a formula, with values in the set of formulae, Rosser and Turquette defined a *generalized quantifier* as any formula of the form:

$$Q(x_1, x_2, \ldots, x_m, F_1, F_2, \ldots, F_t),$$

where $x_1, x_2, \ldots, x_m$ are nominal variables and $F_1, F_2, \ldots, F_t$ formulae built from predicates, nominal and propositional variables, and connectives.

Carnielli (1987) admits a very general class of *distribution quantifiers* defined using multiple-valued matrices as functions mapping subsets of the set of logical values into values. This ingenious construction also directly extends a standard approach to classical quantifiers.

9 Interpretation and Justification

Scholars of the philosophical foundation of logic widely criticized many-valued constructions. The first was the explanation of the logical value 1/2 in Łukasiewicz (1920) resorting to 'future contingents' and a 'possibility' or undetermination of the 0–1 status of propositions. As shown by Gonseth (1941), such interpretation is incompatible with other principles of Łukasiewicz. Whenever α is undetermined, so is $\neg\alpha$ and then $\alpha \wedge \neg\alpha$ is undetermined. That contradicts our intuition since, independently of α's content, $\alpha \wedge \neg\alpha$ is false. The upshot discovers that Łukasiewicz interpretation neglects the mutual dependence of some 'possible' propositions.

Haack (1978) analyses Łukasiewicz's way of avoiding the fatalist conclusion derived from the assumption that the contingent statement 'I shall be in Warsaw at noon on 21 December of the next year' is either true or false in advance of the event. She remarks that this way of rejecting bivalence is wrong, since it depends on a modal fallacy of arguing from "It is necessary that (if α, then β)" to " If α, then it is necessary that β." Urquhart (1986) sees the third logical value as the set $\{0,1\}$ of two 'potential' classical values of a future contingent sentence and defines the implication as getting all possible values of implication. Thus, for example the implication having 0 as antecedent always takes value 1, the implication from 1 to $\{0,1\}$ takes $\{0,1\}$ as the value and the implication from $\{0,1\}$ to $\{0,1\}$ has the value $\{0,1\}$. The last point is inconsistent with the Łukasiewicz stipulation, since the output has to be 1. Therefore, Urquhart claims, the Łukasiewicz table is wrong. It may be of interest that the connective derived by Urquhart is the Kleene strong implication.

Reichenbach (1944) argued that adoption of three-valued logic would provide a solution to some problems raised by quantum mechanics. In order to avoid 'causal anomalies,' Reichenbach presents an extended version of the Łukasiewicz logic, adding further negation and implication connectives. He refers to the third logical value as 'indeterminate' and assigns it to anomalous statements of quantum mechanics. The weak point of Reichenbach's proposal is that certain laws are also classified as 'indeterminate', such as for example, the principle of energy.

The mathematical probability calculus in its simplest form resembles many-valued logic. Łukasiewicz, before 1918, invented a concept of *logical probability* referring to propositions and not to events, see Łukasiewicz (1913). The continuators tried to create a many-valued logic within which logical probability could find a satisfactory interpretation, see, for example, Zawirski (1934), Reichenbach (1935). The Reichenbach–Zawirski theory is based on the assumption that there is a function *Pr* ranging over the set of propositions of a given *standard* propositional language, with values from the real interval [0,1], such that

P1 $0 \leq pr(p) \leq 1$
P2 $Pr(p \vee \neg p) = 1$
P3 $Pr(p \vee q) = Pr(p) + Pr(q)$ if p and q are mutually exclusive ($Pr(p \in q) = 0$)
P4 $Pr(p) = Pr(q)$ when p and q are logically equivalent.

Such probability, however, does not fit any ordinary extensional many-valued logic. Identifying the logical value $v(p)$ with the $Pr(p)$ for $Pr(p) = 1/2$ from P2 and P3 we would, for example, get that

$$1/2 \vee 1/2 = Pr(p \vee \neg p) = 1 \quad \text{and} \quad 1/2 \vee 1/2 = Pr(p \vee p) = Pr(p) = 1/2.$$

A very convincing interpretation of the $\mathcal{N}_0$-valued Łukasiewicz logic of Giles (1974) is based on a dispersive physical interpretation of standard logical language. Each prime proposition in a physical theory is associated through the rules of interpretation with a certain experimental procedure terminating in one of the two possible outcomes, 'yes' and 'no.' The tangible meaning of a proposition is related to the observers and expressed in terms of probability. In the case of prime propositions it is determined from the values of probability of success ascribed by observers in respective experiment, whereas in the case of compound propositions it is determined from the rules of obligation formulated in the nomenclature of *dialogue logic*. The inductive clauses for the connectives, and later for quantifiers, translated back to subjective probability function *pr* conform to the original Łukasiewicz definitions. The set of tautologies of the dialogue logic, that is of formulas to which any valuation assigns non-positive risk-value, coincides with the set of tautologies of the infinite-valued Łukasiewicz logic.

Elimination of the Russell paradox was among the expectations of Łukasiewicz and Bochvar. An interesting work on Łukasiewicz logics related to the question of the unlimited consistency of the comprehension axiom, that is a first-order formula with $\in$ stating the existence of all sets bearing logically expressible properties, was done. It started with Moh Shaw Kwei's (1954) result on the impossibility of the use of finite systems for the purpose, and continued in the 1960s after Skolem (1957) put forward a hypothesis that *CA* was consistent in $\mathcal{N}_1$-valued Łukasiewicz logic. Though several interesting results have been obtained, the question, in its full generality, still remains open.

Scott (1973) replaces many logical values by many valuations using the truth t and falsity f. A definite number of bivalent valuations generates a partition of the set of propositions into types (indexes) corresponding to the original logical values – Scott refers to them as to *indexes*. An n-element set of valuations can thus induce maximally 2^n types. The actual number of types depends on limiting conditions imposed on valuations. An accurate choice of these conditions leads to a relatively simple characterization of the connectives of the logic under consideration. Applying his method, Scott gets a description of the n-valued Łukasiewicz negation and implication connectives through an $(n-1)$-element set of valuations $\{v_0, v_1, \ldots, v_{n-2}\}$. The equalities of the form '$v_i(\alpha)$' should be read as '(the statement) α is true to within the degree i.' Consequently, the numbers $0, 1, \ldots, n-2$ stand for *degrees of error in deviation from the truth*. Degree 0 is the strongest and corresponds to 'perfect' truth or no error: all Łukasiewicz tautologies are schemes of the statements having 0 as their degree of error. The measure of error of the Łukasiewicz implication expresses the amount of *shift* of error between the degree of hypothesis and that of the conclusion.

Urquhart (1973) gave an interpretation motivated by the logic of tenses. The core of it is the relation $\vdash$ between natural numbers of $S_n = \{0, 1, \ldots, n-2\}$ and formulas. '$x \vdash \alpha$' expresses that 'α is true at x' satisfies

If $x \vdash \alpha$ and $x \leq y \in S_n$, then $y \vdash \alpha$.

Adopting ⊢ to particular finite-valued logic requires specifying n, the language, and providing recursive conditions which establish the meaning of connectives. Accordingly, each case results in some Kripke-style semantics with finite number of 'reference points' S_n. For Łukasiewicz and Post logics, Urquhart suggests a temporal interpretation: 0 is the present moment and all other points of reference are future moments. A temporal way of understanding Łukasiewicz negation and implication exhibits the sources of difficulties in getting plausibly intuitive interpretation of many-valued Łukasiewicz logic. Urquhart eventually indicates clauses which 'natural' connectives of negation and implication should satisfy.

10 Applications

Perhaps the most natural of all was the use of many-valuedness to the analysis of vagueness, inexactness, and the paradoxes, see for example Williamson (1994). This application finally gave an impetus to fuzzy set theory and, ultimately to the theory of fuzzy logics, see Zadeh (1975). Zadeh (1965) defines a fuzzy set A of a given domain U as an abstract object characterized by generalized characteristic function U_A with values in the real set [0,1]:

$$U_A : U \rightarrow [0,1].$$

The values of U_A are degrees of membership of elements of U to a fuzzy set A. The extreme values denote, respectively, not belonging to A and the entire membership of A. So, an ordinary set is a special fuzzy set, having only 0 and 1 as possible degrees of membership.

Fuzzy sets model inexact predicates appearing in natural languages. The values of generalized characteristic functions are logical values of propositions obtained from the predicates serving as a basis for a given fuzzy set. Consequently, with fuzzy set algebra of fuzzy (sub)sets of a given domain U can be associated with an uncountable many-valued logic. The inclusion and the operations of a (fuzzy) complement $-$, union $\cup$, and intersection $\cap$ are then stated by means of 'generalized' set-theoretic predicate $\in$ and logical constants (implication, negation, disjunction, and conjunction, respectively).

The choice of the basic logic is to a great extent prejudiced. It occurred that the $\mathcal{N}_1$-valued logic of Łukasiewicz is appropriate and it still remains favorite in the field. The early accounts yielded the (first) understanding of the term 'fuzzy logic' as a certain class of infinitely-valued logics, with Łukasiewicz logics in the foreground.

A typical case of modeling an inexact predicate within the above framework is the following attempt of modeling the classical paradox of a *bald man*. Let us take the two following, intuitively acceptable, propositions:

(1) A man with 20,000 hairs on his head is not bald
(2) A man who has one hair less than somebody who is not bald is not bald as well.

Applying the detachment rule 20,000, we get, by (1) and (2), that a man with no hair is not bald either. The paradox will vanish when the logical value of any proposition 'A man with n hair is not bald' is identified with the degree of membership of a man with n hairs to a fuzzy set 'not-bald.' Then, (2) will have a logical value less than 1, say $1 - \varepsilon$, where $\varepsilon > 0$. And, if in basic logic we use Łukasiewicz's implication. Then as a result of 20,000 derivations we will obtain a proposition of the logical value amounting to $1 - 20{,}000\varepsilon$, thus *practically* false.

Zadeh's (1975) conception of a *fuzzy logic* conveyed the belief that thinking in terms of fuzzy sets is a typical feature of human perception. Fuzzy logic identifies predicates with fuzzy subsets of a given universe and logical values with fuzzy subsets of the set of values of the basic logic. The logical values are labeled *linguistic* entities and, similarly as predicates, may be modified by the so-called *hedges*. Finally, the procedure of linguistic approximation compensates for the lack of closure of the object language and the closure of the set of logical values onto logical connectives. Fuzzy logic is now an autonomic discipline. It seeks to formulate several rules of approximate inference.

Zadeh's fuzzy approach has found its place among accepted methods of artificial intelligence, in computer science and steering theory. It confirmed its usefulness due to reliable applications; see Turner (1984).

The use of many-valued matrices to the formalization of intensional functions, the matrix approximation of syntactically founded non-classical logics and the testing of independence of axioms are worth mentioning. The first use was already suggested by Łukasiewicz, who insisted on the formalization of possibility and necessity within the three-valued logic (see Section 2) and several years later proposed a four-valued system of modal logic in Łukasiewicz (1953). This line of approach has been in some way continued since the algebraic interpretations of Łukasiewicz and Post logics incorporated 'modal' functions in a form of the Boolean-valued endomorphisms. However, from the philosophical point of view these finite-valued interpretation of modalities have no particular value (since as already in 1940 Dugundji proved, no reasonable system of modal logic may be finite-valued), the role of their counterparts in Post algebras occurred which were crucial for the Computer Science applications.

Łoś (1948) showed that, under some reasonable assumptions, the formalization of functions of the kind 'John believes that p' naturally leads to the many-valued interpretation of the belief operators within the scope of the classical logic system. The model situation considered is the case of two persons, who do not agree on all the issues, which may be expressed in propositions. One then obtains four possible evaluations in terms of pairs of classical logic values, that is the truth or falsity, which divides the set of all propositions into four types (or classes) ultimately corresponding to non-classical values. The connectives of negation and implication defined 'naturally,' in reference to their classical counterparts in parallel use for every person, also behave classically. Accordingly, we fall in the four-valued version of *CPC*. The shifting of approach onto the case with more persons results in other formal many-valued interpretations of the classical logic with additional operators. Łoś's construction shows that it is possible to get a many-valued interpretation of some special intensional functions simultaneously adhering to the intuition of bivalence. Since many-valuedness thus received reflects certain relation of two different arguments, a person and a proposition, it has to be classified as untypical semantic correlate.

The successful use of classical logic and Boolean algebras in switching theory and in computer science became established. The algebraic approach enables the application of several techniques for the analysis, synthesis, and minimalization of multiplex networks. And, as early as the 1950s, interests centered also on possibility of the use of many-valued logics for similar purposes. These interests brought about the birth of several techniques for the analysis and synthesis of electronic circuits and relays based mainly on Moisil's and Posts algebras, see for example Rine (1977). The *practical* switchover of two oppositely oriented contacts positioned in parallel branches of a circuit, which have to change their positions simultaneously is the simplest possible electronic circuit to consider within a three-valued framework. Namely, there are good reasons to drop the idealistic assumption affecting the circuit, for example using relays, would really change the positions of both contacts instantly, that is that the circuit would pass from state 1 to state 0. Then, obviously, we get a third state that might also obtain. A generalization of the outlined construction for the case of any number of contacts similarly results in n states. Finally, getting a description of a network composed of such switchovers is performed using Moisil algebras, that is Łukasiewicz n-valued algebras and Post algebras. The most important advantage of the many-valued approach is the possibility of eliminating switching disturbances through the algebraic synthesis of the networks, see, for example, Moisil (1972).

Post algebras found an important application in the systematization of theoretical research concerning programs and higher level programming languages which contain instruction branching programs – the constants $e_0, e_1, \ldots, e_{n-1}$ of Post algebra are then interpreted as devices which keep track of which appropriate branching conditions $W_0, W_1, \ldots, W_{n-1}$. Further to this, Post algebras of order ω^+ form a semantic base for an ω^+-valued extension of algorithmic logic adapted to arbitrary 'wide' branching programs, see Rasiowa (1977).

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